12 Instantons and Gaugino Condensation

12.1 Review of Instantons

Recall that instantons are Euclidean solutions of $D^\mu F^a_{\mu\nu}$, characterized by a size ρ, which approach

$$A_\mu(x) \to iU(x)\partial_\mu U(x)^\dagger \tag{12.1}$$

as $|x| \to \infty$. Instantons break axial $U(1)$ symmetries. Consider the axial symmetry that has charge $+1$ for all (left-handed) fermions. We have

$$\partial_\mu J^\mu_A \propto F^{a\mu\nu}\widetilde{F}^a_{\mu\nu} \tag{12.2}$$

Integrating this current in the instanton background one finds:

$$\int d^4x \partial_\mu J^\mu_A = n\left[\sum_r n_r\, 2T(r)\right] \tag{12.3}$$

thus instantons can create or annihilate fermions. Also an axial rotation of the fermions

$$\psi \to e^{i\alpha}\psi \tag{12.4}$$

is equivalent to a shift of $\theta_{\rm YM}$

$$\theta_{\rm YM} \to \theta_{\rm YM} - \alpha\sum_r n_r\, 2T(r) \tag{12.5}$$

In the instanton background the gauge covariant derivative can be diagonalized,

$$\overline{\sigma}^\mu D_\mu \psi_i = \lambda_i \psi_i \tag{12.6}$$

For a fermion in representation r one finds $2T(r)$ zero eigenvalues. (In the anti-instanton background $\psi^\dagger$ has $2T(r)$ zero eigenvalues.) Consider a fundamental of $SU(N)$ with $T(\Box) = \frac{1}{2}$. We can write ψ_i in terms of a grassman variable and a complex eigenfunction $\psi_i = \xi_i f_i(x)$. We then have

$$\psi = \xi_0 f_0 + \sum_i \xi_i f_i \tag{12.7}$$

$$\psi^\dagger = \sum_i \xi_i^\dagger f_i^* \tag{12.8}$$

where f_0 corresponds to the zero eigenvalue. The path integration over this particular fermion is then

$$\int \mathcal{D}\psi \mathcal{D}\psi^\dagger = \int d\xi_0 \int \prod_{ij} d\xi_i d\xi_j^\dagger \tag{12.9}$$

So

$$\begin{aligned}
\int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp\left(-\int \psi^\dagger \overline{\sigma}^\mu D_\mu \psi\right) \qquad &= \int d\xi_0 \int \prod_{ij} d\xi_i d\xi_j^\dagger \exp\left(-\sum_n \lambda_n \xi_n^\dagger \xi_n\right) \\
&= \int d\xi_0 \int \prod_{ij} d\xi_i d\xi_j^\dagger \prod_n \left(1 - \lambda_n \xi_n^\dagger \xi_n\right) \\
&= \int d\xi_0 \prod_n \ \lambda_n = 0 \ .
\end{aligned} \tag{12.10}$$

However

$$\int \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp\left(-\int \psi^\dagger \overline{\sigma}^\mu D_\mu \psi\right) \psi(x) = \prod_n \ \lambda_n f_0(x) \tag{12.11}$$

At distances much larger than the instanton size 't Hooft showed that instantons produce effective interactions

$$\mathcal{L}_{\text{inst}} = a \det \overline{Q}^{i\alpha} Q_{\alpha j} + h.c. \tag{12.12}$$

This interaction respects the non-Abelian $SU(F) \times SU(F)$ flavor symmetry but breaks the $U(1)_A$ symmetry.

In a theory with scalars that carry gauge quantum numbers, vev's of the scalars prevent us from finding solutions of the classical Euclidean eq. of motion. However we can find approximate solutions when we drop the scalar contribution to the gauge current:

$$D^\mu F_{\mu\nu}^a = 0 \tag{12.13}$$

$$D^\mu D_\mu \phi^j + \frac{\partial V(\phi)}{\partial \phi_j^*} = 0 \tag{12.14}$$

As $|x| \to \infty$

$$A_\mu(x) \to i U(x) \partial_\mu U(x)^\dagger \tag{12.15}$$

$$\phi^j \to U(x) \langle \phi^j \rangle \tag{12.16}$$

Where $\langle \phi^j \rangle$ is a vacuum solution. For small ($\rho < 1/(gv)$) instantons with a completely broken gauge symmetry we find Euclidean actions:

$$S_{\text{inst}} = \frac{8\pi}{g^2} \tag{12.17}$$

$$S_\phi = 8\pi^2 \rho^2 v^2 \tag{12.18}$$

Integrating over instanton locations and sizes we find

$$\begin{aligned} &\int d^4x_0 \int \frac{d\rho}{\rho^5} e^{-S_{\text{inst}} - S_\phi} \\ &= \int d^4x_0 \int \frac{d\rho}{\rho^5} \left(\rho\Lambda\right)^b e^{-8\pi^2\rho^2 v^2} \end{aligned} \tag{12.19}$$

which is dominated at

$$\rho^2 = \frac{b}{16\pi^2 v^2} \tag{12.20}$$

Thus the integration is convergent: breaking the gauge symmetry provides an infrared cutoff.

Note that since A_μ is related to an element of the gauge group at $|x| \to \infty$, the topological character of the instanton relies on

$$U : S^3 \to SU(2) \subset G \tag{12.21}$$

If the scalar fields break the gauge group G down to H, then there will still be pure instanton in the H gauge theory if $SU(2) \subset H$. If the instantons in G/H can be gauge rotated into $SU(2) \subset H$, then all G instanton effects can be accounted for by the effective theory through H instantons. If not, we must add new interactions in the effective theory in order to match the physics properly. Examples of when this is necessary include

$$\begin{aligned} &SU(N) \text{ breaks completely} \\ &SU(N) \to U(1) \\ &SU(N) \times SU(N) \to SU(N)_{\text{diag}} \\ &SU(N) \to SO(N) \end{aligned}$$

This obviously happens whenever there are a different number of zero modes for G and H instantons. In general new interactions must be included in the effective theory when $\pi_3(G/H)$ is non-trivial.

12.2 Gaugino Condensation

Note that in $SU(N)$ SUSY Yang-Mills (with only the only fermion being a gaugino) the $U(1)_R$ symmetry is broken by instantons.

$$\lambda^a \to e^{i\alpha}\lambda^a \tag{12.22}$$

is equivalent to

$$\theta_{\rm YM} \to \theta_{\rm YM} - 2N\alpha \tag{12.23}$$

since λ^a has $2N$ zero-modes in a one-instanton background. This is only a symmetry when

$$\alpha = \frac{k\pi}{N} \tag{12.24}$$

so $U(1)_R$ is broken down to a Z_{2N} subgroup. Assuming that SUSY Yang-Mills has no massless particles, just massive color-singlet composites, then holomorphy and symmetries determine the effective superpotential to be:

$$W_{\rm eff} = a\mu^3 e^{\frac{2\pi i\tau}{N}} \tag{12.25}$$

where

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2(\mu)} \tag{12.26}$$

The gaugino condensate is

$$\begin{aligned}
\langle\lambda^a\lambda^a\rangle &= 16\pi i\frac{\partial}{\partial F_\tau}\ln Z \\
&= 16\pi i\frac{\partial}{\partial F_\tau}\int d^2\theta W_{\rm eff} \\
&= 16\pi i\frac{\partial}{\partial\tau}W_{\rm eff} \\
&= 16\pi i\frac{2\pi i}{N}a\mu^3 e^{\frac{2\pi i\tau}{N}} \\
&= -\frac{32\pi^2}{N}a\Lambda^3
\end{aligned} \tag{12.27}$$

Since

$$\langle\lambda^a\lambda^a\rangle \to e^{2i\alpha}\langle\lambda^a\lambda^a\rangle \tag{12.28}$$

where $\alpha = k\pi/N$, Z_{2N} is broken to Z_2, and there should be N distinct vacua. We see that $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi$ sweeps out N different values for $\langle\lambda^a\lambda^a\rangle$.

References

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